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The vibrations of a free and loaded tyre☆

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Abstract

Using the model of a wheel with a reinforced tyre [Vil'ke VG, Kozhevnikov IF. A model of a wheel with a reinforced tyre. Vestnik MGU. Ser.1. *Matematika Mekhanika* 2004;**4**:37–45], the natural frequencies and natural forms of vibrations of a free or loaded tyre in the neighbourhood of the equilibrium position are determined. The spectrum of natural frequencies and natural forms of vibration are found analytically for a free (unloaded) tyre with a fixed disc. A similar problem is solved numerically in the case of a loaded tyre. The results of this analysis can be used to estimate the level of noise which occurs when a vehicle moves on an uneven surface.

The small vibrations of wheels have been investigated by many researchers. In particular, the vibrations of a tyre have been investigated using the model of an elastic ring and the transmission of vibrations from the road to the axis of the wheel have been analysed.[2] The vibrations of a flexible extensible rotating ring have been considered in the linear formulation[3] and taking into account the geometrical non-linearity.[4] A review of papers devoted to the vibrations of wheels in a complex dynamic vehicle suspension and their influence on the forces transmitted from the tyre to the body of the vehicle can be found in Ref. 5.

1. A free tyre

We will assume that the wheel with the reinforced tyre consists of a disc, which is an absolutely rigid body, having six degrees of freedom, joined to the side wall of the tyre, represented in the undeformed state by parts of two tori and a tread of reinforced inextensible fibres (cords). In the undeformed state the tread is represented by a circular cylinder of radius r. The tyre makes contact with the plane along a part of the tread. The sides possess elastic properties, and their material is described by the Mooney-Rivlin incompressible rubber model.[6]

Suppose the wheel is fixed and unloaded. Then

$$X_1 = \text{const}, \quad X_2 = \text{const}, \quad X_3 = \text{const}, \quad \beta = 0, \quad \kappa = 0, \quad \theta = 0$$

Here X_1, X_2 and X_3 are the coordinates of the centre of mass of the disc C, and β, κ and θ are the swivel, tilt and natural rotation angles respectively. The equations of motion and the condition of the middle line of the tread to be inextensible have the form[1]

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$$-\rho r^3\ddot{u} + n_{11}u'' - n_{01}u + (m_{20} - m_{21})\upsilon' - n_0 + \lambda(1 + u + \upsilon') - [\lambda(u' - \upsilon)]' = 0$$
$$-\rho r^3\ddot{\upsilon} + (m_{02} - m_{12})u' + n_{12}\upsilon'' - n_{02}\upsilon - [\lambda(1 + u + \upsilon')]' - \lambda(u' - \upsilon) = 0 \quad (1.1)$$
$$(1 + u + \upsilon')^2 + (u' - \upsilon)^2 = 1 \rightarrow u \cong -\upsilon'$$

where $u = u(\varphi, t)$ and $\upsilon = \upsilon(\varphi, t)$ are the radial and tangential displacements of points of the middle line of the tread of a moving cylindrical system of coordinates, ρ is the density per unit length of the middle line of the tread, $\lambda = \lambda(\varphi, t)$ is the undetermined Lagrange multiplier (its physical meaning is the tension of the middle line of the tread) and n_0, $n_{01}, n_{11}, n_{02}, n_{12}, m_{02}, m_{20}, m_{12}, m_{21}$ are constant coefficients which depend on the geometrical parameters of the tyre and the pressure inside it (expressions for the coefficients are not given here in view of their length).

In the case of a fixed unloaded wheel, the boundary conditions for the functions $u(\varphi, t)$, and $\upsilon(\varphi, t)$ change into the conditions for these functions to be periodic

$$u(0) = u(2\pi), \quad \upsilon(0) = \upsilon(2\pi), \quad \lambda(0) = \lambda(2\pi)$$

We will represent the Lagrange multiplier in the form

$$\lambda(\varphi, t) = n_0 + \lambda_1(\varphi, t)$$

where λ_1 is a quantity of the first order of smallness. Taking the derivative of both sides of the first equation of (1.1) and adding it to the second, using the linearized condition for the middle line of the tread to be inextensible, we arrive at an equation for the function $\upsilon(\varphi, t)$ and the condition for it to be periodic (a similar equation was previously obtained in[3])

$$\rho r^3\ddot{\upsilon}'' - \rho r^3\ddot{\upsilon} + a_0\upsilon^{(4)} + a_1\upsilon'' + a_2\upsilon = 0, \quad \upsilon(0) = \upsilon(2\pi)$$
$$a_0 = n_0 - n_{11}, \quad a_1 = 2n_0 + n_{01} + n_{12} + m_{20} - m_{21} - m_{02} + m_{12}, \quad a_2 = n_0 - n_{02} \quad (1.2)$$

The natural vibrations of the tyre, described by Eq. (1.2), can be obtained in the form

$$\upsilon = e^{i\omega t}X(\varphi), \quad u = -\upsilon' = -e^{i\omega t}X'(\varphi)$$

where ω is the angular frequency. Then

$$a_0X^{(4)} + (a_1 - \rho r^3\omega^2)X'' + (a_2 + \rho r^3\omega^2)X = 0 \quad (1.3)$$

We will represent the solution of this equation in the form

$$X(\varphi) = G_1e^{p_1\varphi} + G_2e^{p_2\varphi} + G_3e^{-p_1\varphi} + G_4e^{-p_2\varphi} \quad (1.4)$$

where p_i $(i = 1, 2)$ are the roots of the characteristic equation

$$a_0p^4 + (a_1 - \rho r^3\omega^2)p^2 + (a_2 + \rho r^3\omega^2) = 0 \quad (1.5)$$

i.e.

$$p_{1,2}^2 = \left[\rho r^3\omega^2 - a_1 \pm \sqrt{(a_1 - \rho r^3\omega^2)^2 - 4a_0(a_2 + \rho r^3\omega^2)}\right]/(2a_0) \quad (1.6)$$

Since the function $X(\varphi)$ must be 2π-periodic, it follows that only exponents with pure imaginary indices, the modulus of which is a multiple of 2π, need be retained, i.e.

$$p = 2\pi in, \quad n = 0, \pm 1, \pm 2, \ldots$$

In Fig. 1 we show graphs of p_1^2, p_2^2 against $q = -\rho r^3\omega^2/a_0 > 0$ for $c = 22.62$ cm and $a = 0$ cm (on the left) and $a = -5$ cm (on the right); the parameters a and c define the centre of the circle, the arc of which is the side of the tyre. It can be

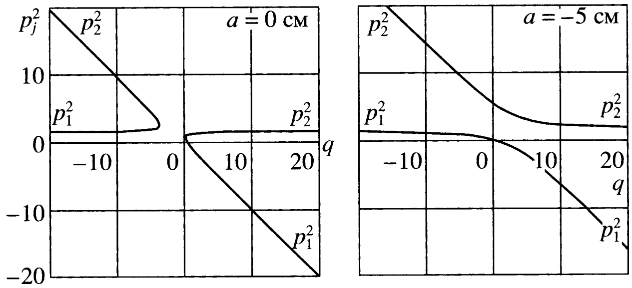

Fig. 1.

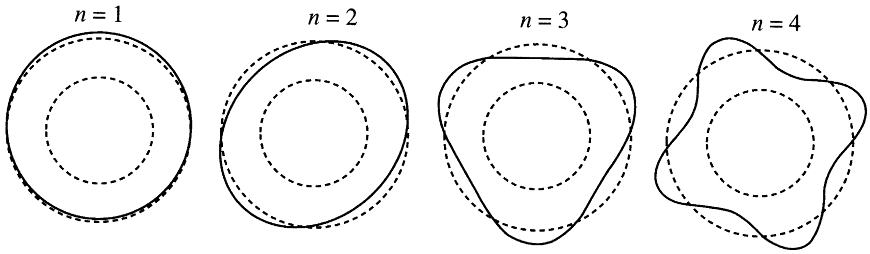

Fig. 2.

seen that only p_1^2 when $p_1 = 2\pi in$ can give an infinite frequency spectrum

$$\omega_n^2 = \frac{-16\pi^4 a_0 n^4 + 4\pi^2 a_1 n^2 - a_2}{\rho r^3 (1 + 4\pi^2 n^2)}, \quad n = 0, 1, 2, \ldots$$

Note that $a_0 < 0$.

The corresponding natural forms are represented by the expression

$$X_n(\varphi) = G_1^* \cos(n\varphi) + G_2^* \sin(n\varphi), \quad \forall G_1^*, G_2^*$$

Any natural form, corresponding to the frequency ω_n, is represented by a linear combination of $\cos n\varphi$ and $\sin n\varphi$. The first four natural forms are shown in Fig. 2, and the corresponding frequencies for the parameters $a = 0$ cm and $c = 22.62$ cm are as follows:

$$\omega_1 = 33.39 \text{ s}^{-1}, \quad \omega_2 = 66.85 \text{ s}^{-1}, \quad \omega_3 = 100.3 \text{ s}^{-1}, \quad \omega_4 = 133.75 \text{ s}^{-1}$$

The coefficient a_0 depends on the pressure in the tyre and its geometrical parameters (a and c). For certain values of these parameters it is impossible to construct the side wall of the tyre, since it must be joined to the disc and the tread by an arc of a circle. This range of values of the parameters (a, c) corresponds to region A in Fig. 3. In the range of values of the parameter B_2 it is possible to construct the side wall of the tyre, but in this case the coefficient $a_0 > 0$, which indicates the presence of solutions of the form $e^{\lambda t} X(\varphi)$, $\lambda > 0$, describing small vibrations of the tyre and, of

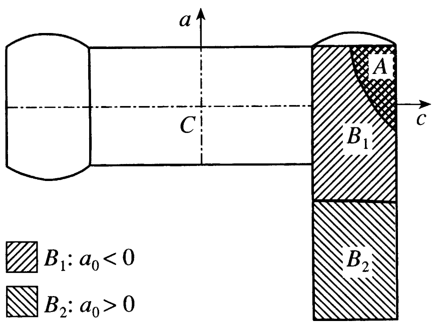

Fig. 3.

course, instability of the chosen shape of the side wall of the tyre. In the regions B_1, the parameters (a, c) are such that for values of these one obtains the required shape of stable side walls of the tyre ($a_0 < 0$).

2. A loaded tyre

Consider the problem of small vibrations of a tyre about the equilibrium position. The equilibrium of a loaded tyre was considered in Ref. 1, when

$$X_1 = \text{const}, \quad X_2 = \text{const}, \quad X_3 = \text{const}, \quad \beta = 0, \quad \kappa = 0, \quad \theta = 0$$

and the variables u and v are functions solely of the angle φ. The contact area is constant $\varphi \in [\varphi_1, \varphi_2]$ (in Ref. 1 it was assumed that the contact area is a rectangle of constant width, equal to the width of the tread, and the variable length; the length of the contact area is defined by two functions of time $\varphi_1(t)$ and $\varphi_2(t)$, which are unknown in advance and are found from the equations of motion). Henceforth we will assume that the wheel is fixed, and during the vibrations the functions u, v, λ, φ_1, φ_2 vary. The corresponding equations have the form (1.1), while the boundary conditions are as follows:

$$\begin{aligned}
&\rho r^3[\dot{u}]_k\dot{\varphi}_k - (-1)^k[\lambda(u' - v)]_{l(k)} + n_{11}[u']_k + r\mu_{1k}\cos\vartheta_k = 0\\
&\rho r^3[\dot{v}]_k\dot{\varphi}_k - (-1)^k[\lambda(1 + u + v')]_{l(k)} + n_{12}[v']_k - r\mu_{1k}\sin\vartheta_k = 0\\
&[u]_k = [v]_k = 0, \quad k = 1, 2; \quad \vartheta_k = \theta + \varphi_k
\end{aligned} \tag{2.1}$$

Here $[f(\vartheta)]_k = f(\vartheta_k + 0) - f(\vartheta_k - 0)$ is the jump in the function at the end point of the contact area. The subscript $l(k)$ is understood to indicate the value of the corresponding function at $\vartheta_1 - 0$ for $k = 1$ and at $\vartheta_2 + 0$ for $k = 2$, and μ_{1k} is the component of the reaction of the constraint at the boundary points of the contact area. We will represent the functions defining the deformations of the middle line of the tread, the Lagrange multipliers and the functions defining the contact area, in the following form

$$u(\varphi, t) = U(\varphi) + U_{\text{vib}}(\varphi, t), \quad v(\varphi, t) = V(\varphi) + V_{\text{vib}}(\varphi, t)$$

$$\lambda(\varphi, t) = \lambda^\circ(\varphi) + \lambda_{\text{vib}}(\varphi, t)$$

$$\mu_{1k}(\varphi, t) = \mu_{1k}^\circ(\varphi) + \mu_{\text{vib}1k}(\varphi, t), \quad \varphi_k(t) = \varphi_k^\circ + \varphi_{\text{vib}k}(t)$$

where U, V, λ°, μ_{1k}°, φ_k° is the solution describing the equilibrium position of the tyre and which satisfies the following equations and boundary conditions

$$\begin{aligned}
&n_{11}U'' - n_{01}U + (m_{20} - m_{21})V' - n_0 + \lambda^\circ(1 + U + V') - [\lambda^\circ(U' - V)]' = 0\\
&(m_{02} - m_{12})U' + n_{12}V'' - n_{02}V - [\lambda^\circ(1 + U + V')]' - \lambda^\circ(U' - V) = 0\\
&(1 + U + V')^2 + (U' - V)^2 = 1 \to U = -V'\\
&-(-1)^k[\lambda^\circ(U' - V')]_{l(k)} + n_{12}[V']_k + r\mu_{1k}^\circ\left(\frac{\pi}{2} - \varphi_k^\circ\right) = 0\\
&-(-1)^k[\lambda^\circ(1 + U + V')]_{l(k)} + n_{12}[V']_k - r\mu_{1k}^\circ = 0\\
&[U]_k = [V]_k = 0, \quad k = 1, 2
\end{aligned} \tag{2.2}$$

From this system we obtain the functions U, V, λ°, μ_{1k}°, φ_k°, where $\lambda^\circ = n_0 + \lambda_1^\circ$, λ_1° is a term of the first order of smallness.

We obtain the variables U_{vib} and V_{vib}, which define small vibrations of the tyre in the neighbourhood of the equilibrium position, from the following system (here it is necessary to take into account the fact that the Lagrange multipliers λ_{vib}, $\mu_{vib\,1k}$ and also $\varphi_{vib\,k}$ are of the first order of smallness):

$$-\rho r^3\ddot{U}_{vib}+(n_{11}-n_0)U''_{vib}+(n_0-n_{01})U_{vib}+(m_{20}-m_{21}+2n_0)V'_{vib}+\lambda_{vib}=0$$
$$-\rho r^3\ddot{V}_{vib}+(m_{02}-m_{12}-2n_0)U'_{vib}+(n_{12}-n_0)V''_{vib}+(n_0-n_{02})V_{vib}-\lambda'_{vib}=0$$
$$(1+U_{vib}+V'_{vib})^2+(U'_{vib}-V_{vib})^2=1\to U_{vib}=-V'_{vib}$$
$$-(-1)^k[n_0(U'_{vib}-V_{vib})]_{l(k)}+n_{11}[U'_{vib}]_k-r\mu^\circ_{1k}\varphi_{vib k}=0 \tag{2.3}$$
$$-(-1)^k[n_0(U_{vib}+V'_{vib})+\lambda_{vib}]_{l(k)}+n_{12}[V'_{vib}]_k-r\mu_{vib1k}=0$$
$$[U_{vib}]_k=[V_{vib}]_k=0,\quad k=1,2$$

Taking the derivative of both sides of the first equation and adding it to the second, using the linearized condition of the middle line of the tread to be inextensible, we obtain the following equation and boundary conditions

$$\rho r^3\ddot{V}''_{vib}-\rho r^3\ddot{V}_{vib}+a_0V^{(4)}_{vib}+a_1V''_{vib}+a_2V_{vib}=0$$
$$V_{vib}(\varphi^\circ_k+\varphi_{vibk})=0,\quad V'_{vib}(\varphi^\circ_k+\varphi_{vibk})=0;\quad k=1,2 \tag{2.4}$$

We can determine $\varphi_{vib\,k}$ from the following dynamic boundary conditions (the fourth relation in (2.3))

$$\varphi_{vib\ k}=\frac{(-1)^k}{r\mu^\circ_{1k}}(n_0-n_{11})V''_{vib}(\varphi^\circ_k)$$

Hence, the contact area "pulsates" at the same frequency as the frequency V''_{vib}.

However, when determining the frequency of the vibrations of the tyre, the value of the contact area can be taken as constant, since its change determines the correction to the frequency of the second order of smallness in the model chosen. Hence, the boundary conditions in problem (2.4) are equivalent to the following

$$V_{vib}(\varphi^\circ_k)+V'_{vib}(\varphi^\circ_k)\varphi_{vib\ k}\approx V_{vib}(\varphi^\circ_k)=0,\quad V'_{vib}(\varphi^\circ_k)+V''_{vib}(\varphi^\circ_k)\varphi_{vib\ k}\approx V'_{vib}(\varphi^\circ_k)=0$$

Henceforth, for simplicity, we will write φ_k instead of φ°_k.

We will represent the functions, which define the small vibrations, and the Lagrange multipliers in the form

$$V_{vib}=e^{i\omega t}X(\varphi),\quad U_{vib}=-V'_{vib}=-e^{i\omega t}X'(\varphi),\quad \lambda_{vib}=\Delta\lambda e^{i\omega t},\quad \mu_{vib\ 1k}=\Delta\mu_{1k}e^{i\omega t}$$

where ω is the angular frequency. Substituting these expressions into Eq. (2.4) we obtain an equation, similar to (1.3), with a solution which has the form (1.4) and characteristic Eq. (1.5).

The coefficients G_i are found from the boundary conditions

$$G_1e^{p_1\varphi_k}+G_2e^{p_2\varphi_k}+G_3e^{-p_1\varphi_k}+G_4e^{-p_2\varphi_k}=0$$
$$G_1p_1e^{p_1\varphi_k}+G_2p_2e^{p_2\varphi_k}-G_3p_1e^{-p_1\varphi_k}-G_4p_2e^{-p_2\varphi_k}=0;\quad k=1,2 \tag{2.5}$$

The homogeneous system (2.5) has a non-zero solution if its determinant is equal to zero:

$$f(\omega)=2p_1p_2[1-\mathrm{ch}(p_1(\varphi_2-\varphi_1))\mathrm{ch}(p_2(\varphi_2-\varphi_1))]+$$
$$+(p_1^2+p_2^2)\mathrm{sh}(p_1(\varphi_2-\varphi_1))\mathrm{sh}(p_2(\varphi_2-\varphi_1))=0 \tag{2.6}$$

The expressions for the functions $p_1=p_1(\omega)$ and $p_2=p_2(\omega)$ are given by formula (1.6). The spectrum of the natural frequencies $\{\omega_n\}_{n=1}^{\infty}$ is found from the characteristic Eq. (2.6).

As an example, we solved Eq. (2.6) numerically for a tyre with dimensions of 175/70 R13 with an internal pressure $p=2$ atmospheres and values of the Mooney-Rivlin constants $k_1=13\ \mathrm{H/cm^2}$ and $k_2=13\ \mathrm{H/cm^2}$ for three values of the contact area (the load).

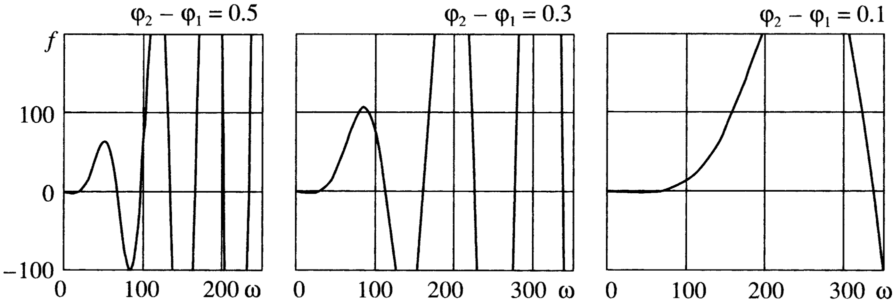

Fig. 4.

In Fig. 4 we show graphs of the function $f(\omega)$ for $a = 0$ cm and $c = 22.62$ cm. The points of intersection of the graph with the abscissa axis give an infinite frequency spectrum. Each natural form X_n, defined by expression (1.4) has its own frequency ω_n (the constants G_i are found from boundary conditions (2.5)). Fig. 4 has three graphs, which show what occurs with the frequencies for contact areas of different value (an increase in the contact area is related to an increase in the external load). It can be seen that an increase in the contact area involves "condensation" of the frequencies. This fact correlates well with the theorem of the theory of small oscillations regarding the behaviour of the natural frequencies when a constraint is imposed.

Eq. (2.6) may give extraneous roots. In particular, it is easy to see that this equation has zero roots: $p_1 = 0$ and $p_2 = 0$. However, the natural forms $X(\varphi)$ corresponding to these cases are equal to zero.

When $p_1 = p_2$ the natural forms must be sought in the form

$$X(\varphi) = (G_1 + G_2\varphi)e^{p_1\varphi} + (G_3 + G_4\varphi)e^{-p_1\varphi} \tag{2.7}$$

The coefficients G_i are found from the boundary conditions

$$\begin{aligned} &(G_1 + G_2\varphi_k)e^{p_1\varphi_k} + (G_3 + G_4\varphi_k)e^{-p_1\varphi_k} = 0 \\ &(G_1p_1 + G_2 + G_2p_1\varphi_k)e^{p_1\varphi_k} + (-G_3p_1 + G_4 - G_4p_1\varphi_k)e^{-p_1\varphi_k} = 0; \quad k = 1, 2 \end{aligned} \tag{2.8}$$

The determinant of system (2.8) can be represented in the form

$$\frac{g(\omega)}{2} = 1 + 2p_1^2(\varphi_2 - \varphi_1)^2 - \operatorname{ch}(2p_1(\varphi_2 - \varphi_1)), \quad p_1^2 = \frac{\rho r^3\omega^2 - a_1}{2a_0} \tag{2.9}$$

The equation $g(\omega) = 0$ has a single root, corresponding to $p_1 = 0$. But in this case we again obtain that $X(\varphi) = 0$.

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References

1. Vil'ke VG, Kozhevnikov IF. A model of a wheel with a reinforced tyre. Vestnik MGU. Ser. 1. *Matematika Mekhanika* 2004;**4**:37–45.
2. Zegelaar PWA, Gong S, Pacejka HB. Tyre Models for the Study of In-Plane Dynamics. 13th IAVSD Symposium of Vehicles on Roads and Tracks, Chengdu, P.R. of China, August 1993. *Vehicle System Dynamics* 1993;**23**(Suppl.):324–42.
3. Zhuravlev VF, Klimov DH. *The Rigid-Body Wave Gyroscope*. Moscow: Nauka; 1985.
4. Vil'ke VG. Analytical Mechanics of Systems with an Infinite Number of Degrees of Freedom. *Pt2 Izd Mekh-Mat Fakul'teta MGU, Moscow* 1997.
5. Willumeit H-P, Böhm F. Wheel vibrations and transient tire forces. *Vehicle System Dynamics* 1995;**24**(6/7):525–50.
6. Oden JT. *Finite Elements in Nonlinear Continua*. New York: McGraw-Hill; 1972.

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